Quantum flag varieties, equivariant quantum $\mathcal{D}$-modules, and localization of Quantum groups.

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Abstract

Let $\mathcal{O}_q(G)$ be the algebra of quantized functions on an algebraic group G and $\mathcal{O}_q(B)$ its quotient algebra corresponding to a Borel subgroup B of G. We define the category of sheaves on the "quantum flag variety of G" to be the $\mathcal{O}_q(B)$-equivariant $\mathcal{O}_q(G)$-modules and prove that this is a proj-category. We construct a category of equivariant quantum $\mathcal{D}$-modules on this quantized flag variety and prove the Beilinson-Bernsteins localization theorem for this category in the case when q is not a root of unity.

1 Introduction

Let k be a field of characteristic zero and fix $q \in k^\star$. Let $\mathfrak{g}$ be a semi-simple Lie algebra over k and let G be the corresponding simply connected algebraic group. Let U_q be a quantized enveloping algebra of $\mathfrak{g}$. Let $\mathcal{O}_q$ be the algebra of quantized functions on G. Let $\mathcal{O}_q(B)$ be the quotient Hopf algebra of $\mathcal{O}_q$ corresponding to a Borel subgroup B of G.

Adopting Grothendieck's philosophy that a space is the same thing as its category of sheaves, we define the "quantized flag variety of G", denoted $\mathcal{M}_{B_q}(G_q)$, to be the category of $\mathcal{O}_q(B)$-equivariant $\mathcal{O}_q$-modules. Thus, an object of $\mathcal{M}_{B_q}(G_q)$ is a left $\mathcal{O}_q$-module M equipped with a right $\mathcal{O}_q(B)$-coaction such that the action map is a morphism of $\mathcal{O}_q(B)$-comodules, see definition 3.1. In this language, the global section functor $\Gamma : \mathcal{M}_{B_q}(G_q) \to k-\text{mod}$ is the functor of taking $\mathcal{O}_q(B)$-coinvariants.

Due to Serre's theorem a projective variety can be described completely algebraically as the quotient of the category of graded modules over a graded ring modulo its subcategory of torsion modules. In particular, the category $\mathcal{M}(G/B)$ of quasi-coherent sheaves on the flag variety G/B is isomorphic to the category $\mathbf{Proj}(\mathcal{O}(G/N))$, where $\mathcal{O}(G/N)$ is the algebra of functions on the basic affine space G/N, N is the unipotent radical of B.

A main idea in the theory of non-commutative geometry, due to Gabriel, Artin and Zhang and others is that this construction generalizes to non-commutative algebras. The algebra $\mathcal{O}(G/N)$ is the so called representation ring of $\mathfrak{g}$ and quantizes naturally to an algebra $\mathcal{O}_q(G/N)$. Lunts and Rosenberg who were the first to study quantized rings of differential operators on flag varieties takes $\mathbf{Proj}(\mathcal{O}_q(G/N))$ as a definition for the category of quantized sheaves on G/B. We prove in proposition 3.5 that our definition is equivalent to theirs. The essential thing to prove is proposition 3.5, which states that $\mathcal{O}_q(\rho)$ is ample. This is not difficult, but much more complicated than the classical case where one simply uses an embedding of G/B into a suitable $\mathbf{P}^n$ (there are many different quantized $\mathbf{P}^n$ and they are not easy to deal with for this purpose). The key ingredients in our proof is Kempf-vanishing of Andersen, Polo and Wen Kexin [APW] and a quantized version of the fact that a space G/N is quasi affine if and only if every rational N-module embeds N-linearly to a rational G-module.

Once this technical difficulty is overcome it turns out that our equivariant sheaves are much easier to deal with than the proj-approach. Actually, except for section 3.6 which concerns bimodule structures on $\mathcal{O}_q(G)$-equivariant sheaves and is independent of the bulk material of this paper, we don't have any explicit need of the proj-category, but we frequently use the fact that $\mathcal{O}_q(\rho)$ is ample.

In particular this becomes evident in the study of D-modules: It is not clear what a quantized ring of differential operators should be on a non-commutative ring (and even less so on a non-commutative space). Lunts and Rosenberg [RL1] gave a definition of such a ring of differential operators, using a definition similar to Grothendieck's classical construction, that works for any graded algebra once they fixed a certain bi-character on it. This construction has the disadvantage that ring of differential operators it produces seems to be too big. They apply this construction to $\mathcal{O}_q(G/N)$ (see [RL2]) and define a D-module on (quantum) G/B to be an object in the quotient category of graded D-modules on G/N modulo torsion modules.

Recently, Tanisaki [T] defined the ring of differential operators on quantum G/N to be the subalgebra of $\mathrm{End}_k(\mathcal{O}_q(G/N))$ generated by $\mathcal{O}_q(G/N)$ and U_q. This is a subalgebra of Lunts and Rosenberg's algebra of differential operators. The category of $\mathcal{D}$-modules he gets on quantum G/B by the proj-construction is equivalent to the one we get.

In our equivariant approach we don't need a ring of differential operators on G/N; we only need a ring $\mathcal{D}_q$ of differential operators on G and we simply define $\mathcal{D}_q$ to be the smash product algebra $\mathcal{O}_q \star U_q$. We define a λ-twisted quantum D-module on G/B (λ is an element in the character group of the weight lattice of $\mathfrak{g}$) to be an object $M \in \mathcal{M}_{B_q}(G_q)$ with an additional action of $\mathcal{D}_q$ such that the coaction of $\mathcal{O}_q(B)$ and the action of $U_q(\mathfrak{b}) \subset U_q \subset \mathcal{D}_q$ on M "differs by λ". The λ-twisted D-modules forms a category denoted $\mathcal{D}^{\lambda}_{B_q}(G_q)$. See definition 4.2.

In the equivariant language, there is no (λ-twisted) sheaf of rings of differential operators on G/B. But we do have a distinguished object $\mathcal{D}^{\lambda}_q$ which represents the global sections. It can be described as the maximal quotient of $\mathcal{D}_q$ that belongs to $\mathcal{D}^{\lambda}_{B_q}(G_q)$. As an object of $\mathcal{M}_{B_q}(G_q)$, $\mathcal{D}^{\lambda}_q$ is isomorphic to the "induced sheaf" $\mathcal{O}_q \otimes M_\lambda$, where M_λ is a Verma module with highest weight λ. As the global section functor Γ is given by $\mathrm{Hom}(\mathcal{D}^{\lambda}_q, \ )$ we see that $\Gamma(\mathcal{D}^{\lambda}_q) = \mathrm{End}(\mathcal{D}^{\lambda}_q)$ is an algebra.

We prove in proposition 4.5 that for each q except a finite set of roots of unity (depending on $\mathfrak{g}$), $\Gamma(\mathcal{D}^{\lambda}_q)$ is isomorphic to U^{fin}_q/J_λ, where J_λ is the annihilator of M_λ.

The proof of 4.5 uses the corresponding classical result for the case $q = 1$ and results of Joseph and Letzter [JL2] which states that the standard filtration on the enveloping algebra $U(\mathfrak{g})$ has a quantized version where the subquotients have the same dimensions as in the classical case.

The main result of this paper is theorem 4.7 which is the quantized version of Beilinson-Bernsteins localization, [BB]. It states that the global section functor gives an equivalence between $\mathcal{D}^{\lambda}_{B_q}(G_q)$ and the category of modules over the algebra U^{fin}_q/J_λ when λ is dominant and regular. Here U^{fin}_q denotes the ad-finite part of U_q. This theorem holds only if q is not a root of unity and the reason for this is that Harish Chandra's description of the center of U_q doesn't hold at a root of unity. Our proof of the localization theorem is almost identical to the one given in [BB].

Lunts and Rosenberg [RL2] conjectured proposition 4.5 and theorem 4.7 for their D-modules and Tanisaki proved them for his.

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2 Generalities

2.1 Quantum groups

See Chari and Pressley [CP] for details about the topics in this section: Let k be a field of characteristic zero and fix $q \in k^\star$. Let $\mathfrak{g}$ be a semi-simple Lie algebra and let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra contained in a Borel subalgebra of $\mathfrak{g}$. Let $P \subset \mathfrak{h}^\star$ be the weight lattice and P_+ the positive weights; the i'th fundamental weight is denoted by ω_i and ρ denotes the half sum of the positive roots. Let $Q \subset P$ be the root lattice and $Q_+ \subset Q$ those elements which have non-negative coefficients with respect to the basis of simple roots. Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$. We let $< , >$ denote a $\mathcal{W}$-invariant bilinear form on $\mathfrak{h}^\star$ normalized by $< \gamma, \gamma >= 2$ for each short root γ.

Let $T_P = \mathrm{Hom}_{groups}(P, k^\star)$ be the character group of P. be the character group of P with values in k (we use additive notation for this group). If $\mu \in P$, then $< \mu, P > \subset \mathbb{Z}$ and hence we can define $q^\mu \in T_P$ by the formula $q^\mu(\gamma) = q^{<\mu,\gamma>}$, for $\gamma \in P$. If $\mu \in P, \lambda \in T_P$ we write $\mu + \lambda = q^\mu + \lambda$. Note that the Weyl group naturally acts on T_P.

Let U_q be the simply connected quantized enveloping algebra of $\mathfrak{g}$ over k. Recall that U_q has algebra generators $E_\alpha, F_\alpha, K_\mu$, α, β are simple roots, $\mu \in P$ subject to the relations

$$K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1,$$

$$K_\mu E_\alpha K_{-\mu} = q^{<\mu,\alpha>} E_\alpha, \quad K_\mu F_\alpha K_{-\mu} = q^{-<\mu,\alpha>} F_\alpha,$$

$$[E_\alpha,\, F_\beta] = \delta_{\alpha,\beta} \frac{K_\alpha - K_{-\alpha}}{q - q^{-1}}$$

and certain Serre-relations that we do not recall here. (We assume that $q^2 \neq 1$.)

Let G be the simply connected algebraic group with Lie algebra $\mathfrak{g}$, B be a Borel subgroup of G and $N \subset B$ its unipotent radical. Let $\mathfrak{b} = \mathrm{Lie}\, B$

and $\mathfrak{n} = \operatorname{Lie} N$ and denote by $U_q(\mathfrak{b})$ and $U_q(\mathfrak{n})$ the corresponding subalgebras of U_q. Then $U_q(\mathfrak{b})$ is a Hopf algebra, while $U_q(\mathfrak{n})$ is only an algebra. Let $\mathcal{O}_q = \mathcal{O}_q(G)$ be the algebra of matrix coefficients of finite dimensional type-1 representations of U_q. There is a natural pairing $(\ ,\ ) : U_q \otimes \mathcal{O}_q \to k$. This gives a U_q-bimodule structure on $\mathcal{O}_q$ as follows

$$ua = a_1(u, a_2), \ \ au = (u, a_1)a_2, \ u \in U_q, a \in \mathcal{O}_q \tag{2.1}$$

Then $\mathcal{O}_q$ is the (restricted) dual of U_q with respect to this pairing. We let $\mathcal{O}_q(B)$ and $\mathcal{O}_q(N)$ be the quotient algebras of $\mathcal{O}_q$ corresponding to the subalgebras $U_q(\mathfrak{b})$ and $U_q(\mathfrak{n})$ of U_q, respectively, by means of this duality. Then $\mathcal{O}_q(B)$ is a Hopf algebra and $\mathcal{O}_q(N)$ is only an algebra.

Verma modules: For each $\lambda \in T_P$ there is the one dimensional $U_q(\mathfrak{b})$-module k_λ which is given by extending λ to act by zero on the E_α's. The Verma-module M_λ is the U_q-module induced from k_λ. Thus M_λ is a cyclic left U_q-module with a generator 1_λ subject to the relations

$$E \cdot 1_\lambda = 0, \ K_\alpha \cdot 1_\lambda = \lambda(\alpha) \cdot 1_\lambda. \tag{2.2}$$

Let $\mu \in P$. We write $k_\mu = k_{q^\mu}$ and $M_\mu = M_{q^\mu}$ in this case. Note that k_μ integrates to an $\mathcal{O}_q(B)$-comodule: we can think of μ as living in the restricted dual of $U_q(\mathfrak{b})$ (i.e. in $\mathcal{O}_q(B)$) and μ is grouplike. The comodule action on k_μ is now given by

$$1_\mu \to 1_\mu \otimes \mu. \tag{2.3}$$

Each 1-dimensional $\mathcal{O}_q(B)$-comodule is isomorphic to k_μ for some $\mu \in P$.

Harish Chandra homomorphism: Let $\mathcal{Z}$ denote the center of U_q. Assume that q is not a root of unity. Given $\lambda \in T_P$ there is the central character $\chi_\lambda : \mathcal{Z} \to k$; it is characterized by the property that $\operatorname{Ker} \chi_\lambda \cdot M_{\lambda-\rho} = 0$. We have $\operatorname{Ker} \chi_\lambda = \operatorname{Ker} \chi_{w\lambda}$.

Let $\lambda \in T_P$. If q is not a root of unity, we say that

- λ is dominant if $\chi_\lambda \neq \chi_{\lambda+\phi}$ for each $\phi \in Q_+ \setminus \{0\}$.
- λ is regular dominant if for all $\phi \in P_+$ and all weights ψ of V_ϕ, $\phi \neq \psi$, we have $\chi_{\lambda+\phi} \neq \chi_{\lambda+\psi}$. (Here V_ϕ is the irreducible finite dimensional type-1 representation of U_q with highest weight ϕ. See also definition 3.6.)

If $\lambda = q^\mu, \mu \in P$ this is equivalent to saying that μ is dominant, respectively regular dominant, in the usual sense.

Finite part of U_q. The algebra U_q acts on itself by the adjoint action $\mathrm{ad} : U_q \to U_q$ where $\mathrm{ad}(u)(v) = u_1 v S(u_2)$. Let U_q^{fin} be the finite part of U_q with respect to this action:

$$U_q^{fin} = \{v \in U_q; \dim \mathrm{ad}(U_q)(v) < \infty\}.$$

This is a subalgebra. (See [JL1].)

We shall frequently refer to a right (resp. left) $\mathcal{O}_q$-comodule as a left (resp. right) G_q-module, etc. If we have two right $\mathcal{O}_q$-comodules V and W, then $V \otimes W$ carries the structure of a right $\mathcal{O}_q$-comodule via the formula

$$\delta(v \otimes w) = v_1 \otimes w_1 \otimes v_2 w_2$$

We shall refer to this action as the *tensor* or *diagonal* action. A similar formula exist for left comodules.

2.2 Proj-categories

We shall use a multigraded version of the classical result about Proj-categories that is basically due to Serre. We consider tuples of data $(\mathcal{C}; \mathcal{O}; s_1, \dots s_l)$ where $\mathcal{C}$ is an abelian category, $\mathcal{O}$ a fixed object of $\mathcal{C}$, $s_1, \dots s_l$ a set of pairwise commuting autoequivalences of $\mathcal{C}$. For $\mathbf{n} = (n_1, \dots, n_l) \in \mathbb{N}^l$, and $M \in \mathrm{Ob}(\mathcal{C})$ we define "twisting-functors" on $\mathcal{C}$ by

$$M(\mathbf{n}) = s_1^{n_1} \cdots s_l^{n_l}(M).$$

We define for any $M \in \mathrm{Ob}(\mathcal{C})$ its global sections $\Gamma(M) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, F)$. We also put $\underline{\Gamma}(M) = \oplus_{\mathbf{n} \in \mathbb{N}^l} \Gamma(M(\mathbf{n}))$.

For any $\mathcal{Z}^l$-graded algebra $R = \oplus_{\mathbf{n} \in \mathcal{Z}^l} R_{\mathbf{n}}$ we denote by $\mathbf{Proj}(R)$ the quotient category of the category of $\mathbb{N}^l$-graded left R-modules modulo the Serre subcategory of torsion object. Here, an object is called torsion if each of its elements is annihilated by $R_{\geq k} = \oplus_{n_1, \dots, n_l \geq k} R_{\mathbf{n}}$ for some $k \geq 0$. Let $\mathcal{C}^0$ denote the set of noetherian objects in $\mathcal{C}$. Artin and Zhang [AZ] proved the following result

Proposition 2.1 *Assume that i) $\mathcal{O}$ is in $\mathcal{C}^0$;*

ii) $\underline{\Gamma}(\mathcal{O})$ is a left-noetherian ring and $\underline{\Gamma}(M)$ is finitely generated over $\Gamma(\mathcal{O})$ for $M \in \mathcal{C}^0$;

iii) For each $M \in \mathcal{C}^0$ there is a surjection $\oplus_{j=1}^{p} \mathcal{O}(-\mathbf{n}_j) \to M$; and

iv) if $M, N \in \mathcal{C}^0$ and $M \to N$ is a surjection, then $\Gamma(M(\mathbf{n})) \to \Gamma(N(\mathbf{n}))$ is surjective for $\mathbf{n} >> 0$. Then $\mathcal{C}$ is equivalent to the category $\mathbf{Proj}(\underline{\Gamma}(\mathcal{O}))$.

We will refer to an autoequivalence satisfying *iii*) and *iv*) as ample.

3 Quantum flag variety

3.1

The composition

$$\mathcal{O}_q \to \mathcal{O}_q \otimes \mathcal{O}_q \to \mathcal{O}_q \otimes \mathcal{O}_q(B) \tag{3.1}$$

defines a right $\mathcal{O}_q(B)$-comodule structure on $\mathcal{O}_q$. A B_q-equivariant sheaves on G_q is a triple (F, α, β) where F is a vector space, $\alpha : \mathcal{O}_q \otimes F \to F$ a left $\mathcal{O}_q$-module action and $\beta : F \to F \otimes \mathcal{O}_q(B)$ a right $\mathcal{O}_q(B)$-comodule action such that α is a right comodule map, where we consider the tensor comodule structure on $\mathcal{O}_q(G) \otimes F$.

Definition 3.1 *We denote $\mathcal{M}_{B_q}(G_q)$ to be the category of B_q-equivariant sheaves on G_q. Morphisms in $\mathcal{M}_{B_q}(G_q)$ are those compatible with all structures.*

Remark 3.2 *In the classical case, when $q = 1$, the category $\mathcal{M}_B(G)$ is equivalent to the category $\mathcal{M}(G/B)$ of quasi-coherent sheaves on G/B.*

We similarly have categories $\mathcal{M}(G_q) := \mathcal{M}_{\{e\}}(G_q) =$ category of $\mathcal{O}_q$-modules (where $\{e\}$ is the one-point group) and $\mathcal{M}_{B_q} := \mathcal{M}_{B_q}(\mathrm{pt}) = B_q$-modules (where pt is the one-point space), $\mathcal{M} := \mathcal{M}(\mathrm{pt}) = k - \mathrm{mod}$.

3.2

We have a *basic diagram* that will be used throughout this paper

$$\begin{array}{ccc} \mathcal{M}(G_q) & \xrightarrow{p} & \mathcal{M} \\ \downarrow_{\pi} & & \downarrow_{\bar{\pi}} \\ \mathcal{M}_{B_q}(G_q) & \xrightarrow{\bar{p}} & \mathcal{M}_{B_q} \end{array} \tag{3.2}$$

Here each arrow denotes a pair of adjoint functors; hence the adjoint pair of functors corresponding to an arrow f will be denoted $(f^\star,\ f_\star)$ and $f_\star$ goes in the direction of the arrow. Here $\pi_\star = (\ ) \otimes \mathcal{O}_q(B)$, where B_q acts on the second factor and $\mathcal{O}_q$ acts via the tensor action (using that $\mathcal{O}_q(B)$ is a

quotient of $\mathcal{O}_q$); $\pi^\star$ = forget; $p_\star$ = forget and $p^\star = \mathcal{O}_q \otimes (\ )$, where $\mathcal{O}_q$ acts on the first factor.

Similary, $\overline{\pi}_\star = (\ ) \otimes \mathcal{O}_q(B)$, where B_q acts on the second factor; $\overline{\pi}^\star$ = forget; $p_\star$ = forget; $\overline{p}^\star = \mathcal{O}_q \otimes (\ )$ where $\mathcal{O}_q$ acts on the first factor and B_q acts via the tensor action.

The diagram is commutative in the sense of usual commutativity after applying lower star (resp. upper star) to all the arrows. All functors considered are exact; hence all "lower star" morphisms maps injectives to injectives.

We define

Definition 3.3 *Let $\lambda \in P$ and put $\mathcal{O}_q(\lambda) = \overline{p}^\star k_{-\lambda}$. We call $\mathcal{O}_q(\lambda)$ a line bundle.*

For each $M \in \mathcal{M}_{B_q}(G_q)$ and $\lambda \in P$ we put

$$M(\lambda) = M \otimes k_{-\lambda}. \tag{3.3}$$

This is an object in $\mathcal{M}_{B_q}(G_q)$ with the $\mathcal{O}_q$-action on the first factor and the tensor B_q-action called the λ-twist of M.

3.3

Definition 3.4 *The global section functor $\Gamma : \mathcal{M}_{B_q}(G_q) \to k - \mathrm{mod}$ is defined by*

$$\Gamma(M) = \mathrm{Hom}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_q, M) = \{m \in M; \Delta_B(m) = m \otimes 1\}.$$

This is the set of B_q-invariants in M.

We can now state our main result about the category $\mathcal{M}_{B_q}(G_q)$.

Proposition 3.5 1)'*Each object in $\mathcal{M}_{B_q}(G_q)$ is a quotient of a direct sum of $\mathcal{O}_q(\lambda)$'s.*

2) *Any surjection $M \twoheadrightarrow M'$ of noetherian objects in $\mathcal{M}_{B_q}(G_q)$ induces a surjection $\Gamma(M(\lambda)) \twoheadrightarrow \Gamma(M'(\lambda))$ for $\lambda >> 0$.*

Here the notation $\lambda >> 0$ means that $< \lambda, \alpha^\wedge >$ is a sufficiently large integer for each simple root α. Thus, the proposition can be phrased as: $\mathcal{O}_q(\lambda)$ is ample if $< \lambda, \alpha^\wedge >> 0$ for each simple root α.

Definition 3.6 *Let* $V_\lambda = \Gamma(\mathcal{O}_q(\lambda))$ *and let* $A = \oplus_{\lambda\in P_+} V_\lambda$ *be the representation ring of* U_q*.*

Note that the V_λ's, $\lambda \in P_+$ are the simple finite dimensional U_q-modules if q is not a root of unity.

Corollary 3.7 *The category* $\mathcal{M}_{B_q}(G_q)$ *is equivalent to* $\mathbf{Proj}(A)$*.*

Proof of corollary 3.7. Then, with the notations of section 2.2

$$A = \oplus_{\lambda\in P_+}\Gamma(\mathcal{O}_q(\lambda)) = \underline{\Gamma}(\mathcal{O}_q).$$

Hence we are left to show that the conditions $i) - iv)$ of proposition 2.1 are satisfied for the tuple $(\mathcal{M}_{B_q}(G_q), \mathcal{O}_q, s_1, \ldots, s_l)$, where $s_i(M) = M(\omega_i)$ and we recall that the ω_i's are the fundamental weights. Now, $iii) - iv)$ is proposition 3.5; $i)$ holds because $\mathcal{O}_q$ is a noetherian ring and $ii)$ is clearly true for line bundles and then follows for general modules from $iii)$. $\square$

3.4

The following two sections are devoted to the proof of proposition 3.5. Apart from the interesting results corollary 3.10 and lemma 3.14 the proof consists mostly of rather technical standard arguments. In this section we show that various categories have enough injectives and calculate some cohomology groups. We deduce in corollary 3.10 that Kempf-vanishing holds in $\mathcal{M}_{B_q}(G_q)$.

Let $M \in \mathcal{M}_{B_q}(G_q)$. The adjunction map $\pi_\star\pi^\star M \to M$ (which is given by $\mathrm{Id}\otimes$counit) has a splitting given by the comodule action $M \to M\otimes\mathcal{O}_q(B) = \pi_\star\pi^\star M$. Let I be an injective hull of $\pi^\star M$ in $\mathcal{M}(G_q)$. Then M embeds into $\pi_\star I$ and we conclude that

Lemma 3.8 *The category* $\mathcal{M}_{B_q}(G_q)$ *has enough injectives.*

Let $\tilde\Gamma : \mathcal{M}_{B_q} \to k-\mathrm{mod}$ be the functor of taking B_q-invariants on $\mathcal{M}_{B_q}$. Thus derived functors of Γ and $\tilde\Gamma$ are defined. We have

$$\Gamma = \widetilde{\Gamma}\circ\overline{p}_\star \tag{3.4}$$

The category $\mathcal{M}_{B_q}$ has enough injectives because $\overline{\pi}_\star$ maps injectives to injectives, each object in $\mathcal{M}(\mathrm{pt})$ is injective and any $M \in \mathcal{M}_{B_q}$ imbeds to $\overline{\pi}_\star\overline{\pi}^\star M$. We have

Lemma 3.9 1) *If $I \in \mathcal{M}_{B_q}$ is injective then $\overline{p}^\star I$ is Γ-acyclic.* 2) *The functor Γ has finite cohomological dimension and the formula $R\Gamma(M) = R\widetilde{\Gamma}(\overline{p}_\star M)$ holds for $M \in \mathcal{M}_{B_q}(G_q)$.*

Proof 1) Let $I \in \mathcal{M}_{B_q}$ be injective. Then I imbeds to $\overline{\pi}_\star\overline{\pi}^\star(I) = \mathcal{O}_q(B)\otimes I \cong \mathcal{O}_q(B)^{\dim I}$. Since $\overline{\pi}_\star$ preserves injectives and every object in $\mathcal{M}$ is injective, $\overline{\pi}_\star\overline{\pi}^\star(I)$ is injective. Since I is injective this embedding splits. Thus it suffices to prove that $\overline{p}^\star(\mathcal{O}_q(B))$ is Γ-acyclic. We have $\overline{p}^\star(\mathcal{O}_q(B)) = \pi_\star(\mathcal{O}_q)$ and conclude

$$R^j\Gamma(\overline{p}^\star(\mathcal{O}_q(B))) = \mathrm{Ext}^j_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_q, \pi_\star(\mathcal{O}_q)) \cong$$

$$\mathrm{Ext}^j_{\mathcal{M}(G_q)}(\pi^\star(\mathcal{O}_q), \mathcal{O}_q) = \mathrm{Ext}^j_{\mathcal{M}(G_q)}(\mathcal{O}_q, \mathcal{O}_q),$$

where we used that $\pi_\star$ is exact and preserves injectives in the second isomorphism. Since $\mathcal{O}_q$ is projective in $\mathcal{M}(G_q)$ the last term vanishes for $j > 0$.

2) Andersen, Polo and Wen Kexin [APW] has shown that the functor $\Gamma \circ \overline{p}^\star$ has cohomological dimension $\leq \dim G/B$. Let $M \in \mathcal{M}_{B_q}(G_q)$. Since k is a direct summand in $\mathcal{O}_q$, M is a direct summand in $\overline{p}^\star\overline{p}_\star(M) = \mathcal{O}_q \otimes M$ as a B_q-module. Thus, by 1), $R^i\widetilde{\Gamma}(M)$ is a direct summand in $R^i(\Gamma \circ \overline{p}^\star)(\overline{p}_\star(M))$ and the latter module vanishes for $i > \dim G/B$.

Let $M \to I_\bullet$ be an injective resolution in $\mathcal{M}_{B_q}(G_q)$. We get again

$$R^i\Gamma(M) = \mathrm{H}^i(\Gamma(I_\bullet)) = \mathrm{H}^i(\widetilde{\Gamma}(\overline{p}_\star I_\bullet)) = R^i\tilde{\Gamma}(\overline{p}_\star M)$$

and the last term vanishes for $i > \dim G/B$. $\square$

Corollary 3.10 *[Kempf vanishing] For each $\lambda \in P_+$ and each $i > 0$ we have $R^i\Gamma(\mathcal{O}_q(\lambda) = 0$.*

Proof Let $\lambda \in P$. Choose an injective resolution $k_\lambda \to I_\bullet$ in $\mathcal{M}_{B_q}$. Then

$$R^i(\Gamma \circ \overline{p}^\star)(k_\lambda) = \mathrm{H}^i(\Gamma(\overline{p}^\star I_\bullet)) = R^i\Gamma(\mathcal{O}_q(\lambda))$$

since the $\overline{p}^\star I_\bullet$ are Γ-acyclic, by lemma 3.9. Now, it is shown in [APW] that $R^i(\Gamma \circ \overline{p}^\star)(k_\lambda) = 0$ for $i > 0$, if $\lambda \in P_+$. $\square$

3.5

In this section we introduce a G_q-equivariant structure on certain objects in $\mathcal{M}_{B_q}(G_q)$. We prove the key lemma 3.14 and finally we prove proposition 3.5.

Let $V \in \mathcal{M}_{G_q}$. Denote by $V|B_q$ the module V restricted to B_q and by V^{triv} the trivial B_q-module whose underlying space is V. We have the following crucial fact

Lemma 3.11 *The objects $\overline{p}^\star(V|B_q)$ and $\overline{p}^\star(V^{triv})$ are isomorphic in $\mathcal{M}_{B_q}(G_q)$.*

Proof. The map $\overline{p}^\star(V|B_q) \to \overline{p}^\star(V^{triv})$ is given by $a \otimes v \to av_2 \otimes v_1$. It is easily checked that this is an isomorphism. $\square$

For any $V \in \mathcal{M}_{B_q}$, $\overline{p}^\star V$ carries the additional structure of a right G_q-module via the (right) action on the first factor. This structure is compatible with the left $\mathcal{O}_q$-action and makes $\overline{p}^\star V$ a $B_q - G_q$-bimodule. We denote by ${}_{G_q}\mathcal{M}_{B_q}(G_q)$ the category of all objects in $\mathcal{M}_{B_q}(G_q)$ that carry this additional structure. We have

Lemma 3.12 *The functor $\overline{p}^\star$ induces an equivalence $\mathcal{M}_{B_q} \to {}_{G_q}\mathcal{M}_{B_q}(G_q)$.*

Proof. For $M \in {}_{G_q}\mathcal{M}_{B_q}(G_q)$ denote its $\mathcal{O}_q$-comodule action by Δ and let $(M)^{G_q} = \{m \in M; \Delta m = 1 \otimes m\}$ be the set of G_q-invariants. It is straight forward to verify that the functor $(\ )^{G_q}$ is inverse to $\overline{p}^\star$. $\square$

Remark 3.13 *The map $\overline{p}^\star(V|B_q) \to \overline{p}^\star(V^{triv})$ in lemma 3.11 becomes an isomorphism in ${}_{G_q}\mathcal{M}_{B_q}(G_q)$ if we modify the G_q action on $\overline{p}^\star(V^{triv})$: we define the new G_q-action to be the diagonal action.*

We first prove the following

Lemma 3.14 *Assume $V \in \mathcal{M}_{B_q}$ is finite dimensional and satisfies the following: if k_λ is a one dimensional sub quotient of V then $\lambda \in P_+$. Then there is a f.d. $W \in \mathcal{M}_{G_q}$ and an B_q-linear surjection $W \twoheadrightarrow V$.*

Proof of lemma 3.14. We have induction and restriction functors between categories

$$\mathcal{M}_{B_q} \underset{\mathrm{res}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathcal{M}_{G_q} \tag{3.5}$$

Let $V \in \mathcal{M}_{B_q}$. We have $\operatorname{Ind}_{B_q}^{G_q}(V) = (\mathcal{O}_q \otimes V)^{B_q}$. For each one-dimensional sub quotient k_λ of V the adjunction morphism

$$\operatorname{res}_{G_q}^{B_q} \operatorname{Ind}_{B_q}^{G_q}(k_\lambda) = \operatorname{res}_{G_q}^{B_q}(V_\lambda) = V_\lambda \to k_\lambda$$

is surjective, since $\lambda \in P_{++}$. An easy induction using corollary 3.10 shows that the functor $res_{G_q}^{B_q} \operatorname{Ind}_{B_q}^{G_q}$ is exact on any sequence of the form $V' \to V \to V/V'$ for any submodule V' of V. By induction and the five lemma we conclude that $\operatorname{res}_{G_q}^{B_q} \operatorname{Ind}_{B_q}^{G_q}(V) \to V$ is surjective.

We take W to be any f.d. G_q-submodule of $\operatorname{Ind}_{B_q}^{G_q}(V)$ that surjects to V.

Proof of proposition 3.5. 1) Let $M \in \mathcal{M}_{B_q}(G_q)$. We can assume that M is noetherian. Take a minimal set of generators of M as an $\mathcal{O}_q$-module and let V be the B_q-module they generate; V is f.d. by the noetherian hypothesis. We get a surjection $\overline{p}^\star V \twoheadrightarrow M$ in $\mathcal{M}_{B_q}(G_q)$. Take $\lambda \in P$ such that $V \otimes k_\lambda$ satisfies the assumption of lemma 3.14 and let W be a f.d. G_q-module that surjects to $V \otimes k_\lambda$. Then $\mathcal{O}_q \otimes W$ surjects to $(\mathcal{O}_q \otimes V)(\lambda)$ and hence to $M(\lambda)$. It follows from lemma 3.11 that $\mathcal{O}_q \otimes W$ is generated by its B_q-invariants. Hence $M(\lambda)$ is as well, i. e. we have a surjection $\mathcal{O}_q(-\lambda)^m \twoheadrightarrow M$.

2). Let $M \twoheadrightarrow M'$ be a surjection in $\mathcal{M}_{B_q}(G_q)$. Let F_0 be a direct sum of line bundles and $F_0 \twoheadrightarrow M$ a surjection. If we can prove that the composition $F_0 \twoheadrightarrow M'$ induces a surjection $\Gamma(F_0(\lambda)) \to \Gamma(M'(\lambda))$ for suitable λ it will follow that the map $\Gamma(M(\lambda)) \to \Gamma(M'(\lambda))$ is surjective for such λ as well.

Put $n = \dim G/B$ which we recall is the cohomological dimension of the functor Γ and pick a resolution

$$F_n \to \ldots \to F_1 \to F_0 \to M' \to 0 \tag{3.6}$$

where each F_i is a direct sum of line bundles. Let λ be sufficiently large for the following property $(\star)$ to hold: each $F_i(\lambda)$ is a direct sum of various $\mathcal{O}_q(\mu)$, where each $\mu \in P_+$. Tensoring 3.6 with $k_{-\lambda}$ we get an exact sequence

$$F_n(\lambda) \xrightarrow{f_n} \ldots \xrightarrow{f_2} F_1(\lambda) \xrightarrow{f_1} F_0(\lambda) \xrightarrow{f_0} M'(\lambda) \xrightarrow{f_{-1}} 0 \tag{3.7}$$

Put $K_i = \operatorname{Ker} f_i$. We must show that $\Gamma(f_0)$ is surjective. We have short exact sequences $K_i \hookrightarrow F_i(\lambda) \twoheadrightarrow K_{i-1}$ inducing exact sequences

$$R^i\Gamma(F_i(\lambda)) \to R^i\Gamma(K_{i-1}) \to R^{i+1}\Gamma(K_i) \to R^{i+1}\Gamma(F_i(\lambda))$$

By ($\star$) and corollary 3.10, we get isomorphisms $R^i\Gamma(K_{i-1}) \cong R^{i+1}\Gamma(K_i)$, for $i \geq 1$. Now, $R^{n+1}\Gamma(K_n) = 0$, because Γ has cohomological dimension n; hence $R^1\Gamma(K_0) = 0$. Considering the above sequence when $i = 0$ we conclude that $\Gamma(f_0)$ is surjective.

3.6 G_q-commutativity of A

The results in this section are not needed for the rest of this paper.

Classically, a sheaf of $\mathcal{O}_{G/B}$-modules is a bimodule as $\mathcal{O}_{G/B}$ is commutative. In the quantum case this is no longer true. Yet the class of G_q-equivariant objects in $\mathbf{Proj}(A)$ admits an A-bimodule structure. Using corollary 3.7 one deduces that the G_q-equivariant objects in $\mathcal{M}_{B_q}(G_q)$ act on $\mathcal{M}_{B_q}(G_q)$; we suggestively denote this action by $\otimes_{\mathcal{O}_q}$.

We recall the notion of a commutative algebra in a braided tensor category.

Definition 3.15 *Let $\mathcal{B}$ be a braided tensor category. An algebra in $\mathcal{B}$ is a pair (R, m) where $R \in \mathrm{Ob}(\mathcal{B})$ and $m : R \otimes R \to R$ satisfying the usual associativity axiom. R is called commutative if the diagram*

$$\begin{array}{ccc} R \otimes R & \stackrel{m}{\to} & R \\ \downarrow_\sigma & & \| \\ R \otimes R & \stackrel{m}{\to} & R \end{array} \tag{3.8}$$

commutes, where σ is the braiding.

Similarly, one can define left modules over R, etc, in the braided tensor category. If R is commutative in $\mathcal{B}$ then left modules are bimodules: Let M be a left R-module. Composing the left action with the braiding we get a right action

$$M \otimes R \to R \otimes M \to M$$

It is easily verified that this structure commutes with the left structure, giving us the asserted bimodule structure.

We now consider the braided tensor category $U_q-\mathrm{grmod}_P$ of P-graded left U_q-modules (we assume additionally that each braided component is finite dimensional). We assume that q has a square root in k and fix such a root $q^{1/2}$. The braiding in $U_q - \mathrm{grmod}_P$ is the product of the usual braiding on U_q-modules and the braiding on the category of P-graded vector spaces given by the bicharacter $q^{1/2<\deg(\ ),\deg(\ )>}$.

We have the following simple lemma

Lemma 3.16 *The algebra A defined in definition 3.6 is commutative in U_q- grmod_P.*

Proof. The coquasitriangularity of $\mathcal{O}_q$ implies that $\mathcal{O}_q$ is commutative in the category of $U_q \otimes U_q^{\mathrm{op}}$-modules (with the obvious braiding). The subalgebra $A \cong \mathcal{O}_q^{N_q}$ of $\mathcal{O}_q$ is no longer an $U_q \otimes U_q^{\mathrm{op}}$-module, but an object in $U_q-\mathrm{grmod}_P$ and the braiding of $U_q \otimes U_q^{\mathrm{op}}$ acts as the braiding in $U_q - \mathrm{grmod}_P$ making it a commutative algebra there. □

The G_q-equivariant objects in $\mathbf{Proj}(A)$ are by definition those that corresponds to ${}_{G_q}\mathcal{M}_{B_q}(G_q)$ under the equivalence in corrolary 3.7. The following result will be useful in the next section

Corollary 3.17 *Any G_q-equivariant M in $\mathbf{Proj}(A)$ is an A-bimodule.*

Proof. Note that G_q-equivariant objects in $\mathbf{Proj}(A)$ can be thought of as graded A-modules with a compatible $\mathcal{O}_q$-comodule structure. By lemma 3.16 and the previous discussion it follows that they are A-bimodules. □
This way, we get an action

$$ {}_{G_q}\mathcal{M}_{B_q}(G_q) \otimes \mathcal{M}_{B_q}(G_q) \to \mathcal{M}_{B_q}(G_q),\ M \times N \to M \otimes_{\mathcal{O}_q} N \tag{3.9} $$

This suggestive notations indicates (ofcourse) that one can define an $\mathcal{O}_q$-bimodule structure on ${}_{G_q}\mathcal{M}_{B_q}(G_q)$ but we didnt work this out.

4 D-modules on Quantum flag variety

4.1 Ring of differential operators on G_q

Recall the U_q-bimodule structure on $\mathcal{O}_q$ given by 2.1.

Definition 4.1 *We define the ring of quantum differential operators on G_q to be the smash product algebra $\mathcal{D}_q := \mathcal{O}_q \star U_q$. So $\mathcal{D}_q = \mathcal{O}_q \otimes U_q$ as a vector space and multiplication is given by*

$$ a \otimes u \cdot b \otimes v = au_1(b) \otimes u_2 v. \tag{4.1} $$

We consider now the ring $\mathcal{D}_q$ as a left U_q-module, via the left U_q-action on $\mathcal{O}_q$ in 2.1 and the left adjoint action on itself. (This is not the action induced from the ring embedding $U_q \to 1 \otimes U_q \subset \mathcal{D}_q$.) This way $\mathcal{D}_q$ becomes a module algebra for U_q:

$$u \cdot a \otimes v = au_1(b) \otimes u_{21} v S(u_{22}). \tag{4.2}$$

In the following we will use the restriction of this action to $U_q(\mathfrak{b})$. As $U_q(\mathfrak{g})$ is not locally finite with respect to the adjoint action on itself, this $U_q(\mathfrak{b})$-action doesn't integrate to a B_q-action. Thus $\mathcal{D}_q$ is not an object of $\mathcal{M}_{B_q}(G_q)$; however, $\mathcal{D}_q$ has a subalgebra $\mathcal{D}_q^{fin} = \mathcal{O}_q \star U_q^{fin}$ which belongs to $\mathcal{M}_{B_q}(G_q)$. This fact will be used below.

4.2 $\mathcal{D}_q$-modules on flag variety

Let $\lambda \in T_P$.

Definition 4.2 *A (B_q, λ)-equivariant $\mathcal{D}_q$-module is a triple (M, α, β), where M is a k-module, $\alpha : \mathcal{D}_q \otimes M \to M$ a left $\mathcal{D}_q$-action and $\beta : M \to M \otimes \mathcal{O}_q(B)$ a right $\mathcal{O}_q(B)$-coaction. The latter action induces an $U_q(\mathfrak{b})$-action on M also denoted by β. These actions are related as follows:*

i) The $U_q(\mathfrak{b})$-action on $M \otimes k_\lambda$ given by $\beta \otimes \lambda$ and by $(\alpha|_{U_q(\mathfrak{b})}) \otimes \mathrm{Id}$ coincide.
ii) The map α is $U_q(\mathfrak{b})$-linear with respect to the β-action on M and the action on $\mathcal{D}_q$ that is given by 4.2 .

These objects form a category denoted $\mathcal{D}^\lambda_{B_q}(G_q)$. There is the forgetful functor $\mathcal{D}^\lambda_{B_q}(G_q) \to \mathcal{M}_{B_q}(G_q)$. Morphisms in $\mathcal{D}^\lambda_{B_q}(G_q)$ are morphisms in $\mathcal{M}_{B_q}(G_q)$ that are $\mathcal{D}_q$-linear.

We define

Definition 4.3 *$\mathcal{D}_q^\lambda$ is the maximal quotient of $\mathcal{D}_q$ which is an object of $\mathcal{D}^\lambda_{B_q}(G_q)$.*

Thus, a simple computation shows that $\mathcal{D}_q^\lambda \cong \mathcal{D}_q/\mathcal{D}_q I$ where

$$I = \{E_i, K_i - \lambda(K_i);\ 1 \le i \le l\} \tag{4.3}$$

Note that $\mathcal{D}_q I$ is not a two-sided ideal and hence $\mathcal{D}_q^\lambda$ is not a ring. We have

$$\mathcal{D}_q^\lambda = \mathcal{O}_q \otimes (U_q/U_q I) \cong \overline{p}^\star(M_\lambda) \tag{4.4}$$

as a vector space. We define the global section functor $\Gamma : \mathcal{D}^\lambda_{B_q}(G_q) \to \mathcal{M}$ to be the global section functor on $\mathcal{M}_{B_q}(G_q)$ composed with the forgetful functor $\mathcal{D}^\lambda_{B_q}(G_q) \to \mathcal{M}_{B_q}(G_q)$. Thus $\Gamma = (\ )^{B_q}$, is the functor of taking B_q invariants (with respect to the action β) and we obviously have

$$\Gamma = \mathrm{Hom}_{\mathcal{D}^\lambda_{B_q}(G_q)}(\mathcal{D}^\lambda_q,\ ). \tag{4.5}$$

In particular, $\Gamma(\mathcal{D}^\lambda_q) = \mathrm{End}_{\mathcal{D}^\lambda_{B_q}(G_q)}(\mathcal{D}^\lambda_q)$ is a ring with multiplication induced from that in $\mathcal{D}_q$.

4.3

Definition 4.4 *Let M_λ be a Verma module with highest weight λ and put $J_\lambda = \mathrm{Ann}_{U_q^{fin}}(M_\lambda)$. We define $U^\lambda_q = U^{fin}_q / J_\lambda$.*

We have

Proposition 4.5 *There is a ring injection $U^\lambda_q \to \Gamma(\mathcal{D}^\lambda_q)$ which is an isomorphism for all q except a finite set of roots of unity depending on the root data.*

Proof of proposition 4.5. There is the natural surjection $U^\lambda_q \to M_\lambda$. It induces a surjective map

$$\overline{p}^\star(U^\lambda_q) \to \overline{p}^\star(M_\lambda) = \mathcal{D}^\lambda_q \tag{4.6}$$

Since U^λ_q is a G_q-module, $\Gamma(\overline{p}^\star(U^\lambda_q))$ is isomorphic to U^λ_q. Applying Γ to 4.6 we get the map

$$U^\lambda_q \to \Gamma(\mathcal{D}^\lambda_q) \tag{4.7}$$

The map 4.7 is injective when $\lambda = 0$. Let $\mathcal{O}_{q,loc}$ be the localization of $\mathcal{O}_q$ defined by De Concini and Lyubashenko, [DL]. This is an object in $\mathcal{D}^0_{B_q}(G_q)$. Here comes the structures: As a (right) $\mathcal{O}_q(B)$-comodule

$$\mathcal{O}_{q,loc} = \mathcal{O}_q(N) \otimes \mathcal{O}_q(B) \tag{4.8}$$

Thus, the map $\beta : \mathcal{O}_{q,loc} \to \mathcal{O}_{q,loc} \otimes \mathcal{O}_q(B)$ is given by the coproduct of $\mathcal{O}_q(B)$. The map $\alpha : \mathcal{D}_q \otimes \mathcal{O}_{q,loc} \to \mathcal{O}_{q,loc}$ is given as follows: the $\mathcal{O}_q$-module structure on $\mathcal{O}_{q,loc}$ is the natural one coming from the localization. The U_q-action on $\mathcal{O}_q$ extends to the localization.

The restriction to U_q^0 of the $\Gamma(\mathcal{D}_q^0)$-action on $\Gamma(\mathcal{O}_{q,loc})$ comes from the natural right action of U_q^0 on $\mathcal{O}_{q,loc}$. It now follows from 4.8 that $\Gamma(\mathcal{O}_{q,loc})$ is isomorphic to $M_0^\star$ as an U_q^0-module. The injectivity claim now follows since U_q^0 by definition acts faithfully on M_0 and hence on M_0^*.

The map 4.7 is an isomorphism for $\lambda = 0$. We define a $\mathbb{Z}$-filtration on U_q by putting $\deg E_i, \deg F_i = 1$ and $\deg K_i = -1$. Denote by $\mathcal{F}_j(Object)$ the j'th filtered part of a filtered $Object$; the associated graded object is denoted by

$$\mathrm{gr}(Object) = \oplus \, \mathrm{gr}_j(Object) = \oplus \mathcal{F}_j(Object)/F_{j-1}(Object).$$

Intersecting our filtration of U_q with U_q^{fin} we get a filtration on U_q^{fin} satisfying $\mathcal{F}_j(U_q^{fin}) = 0$ for $j < 0$ and $\dim \mathcal{F}_j(U_q^{fin}) < \infty$ for all j. We get (positive) quotient filtrations on M_0 and U_q^0. This way, 4.6 and 4.7 become filtered maps. We get maps

$$\mathcal{F}_j(U_q^0) \to \mathcal{F}_j(\Gamma(\mathcal{D}_q^0)) = \Gamma(\mathcal{F}_j(\mathcal{D}_q^0)) = \Gamma(\overline{p}^\star(\mathcal{F}_j(M_0))) \tag{4.9}$$

and hence maps

$$\mathrm{gr}_j(U_q^0) \to \mathrm{gr}_j(\Gamma(\mathcal{D}_q^0)) \to \Gamma(\mathrm{gr}_j(\mathcal{D}_q^0)) = \Gamma(\overline{p}^\star(\mathrm{gr}_j(M_0))) \tag{4.10}$$

Put

$$\mu_j(q) = \dim_k \mathrm{gr}_j(U_q^0), \ \nu_j(q) = \dim_k \Gamma(\mathrm{Ind}(\mathrm{gr}_j(M_\lambda))).$$

By Kostants theorem ([D], chapter 8) $\mu_j(1) = \nu_j(1)$. By results of Joseph and Letzter [JL2] $\mu_j(q)$ is constant for all q except a finite set of roots of unity. By results of [APW] the k-dimension of the global sections of the induction of a finite dimensional B_q-module does not depend on q. Hence, $\nu_j(q)$ is independent of q. Hence 4.10 is an isomorphism for each j. Hence, 4.7 with $\lambda = 0$ is an isomorphism by standard arguments.

The map 4.7 is an isomorphism for general λ. We have filtrations on U_q^λ, M_λ etc and get maps corresponding to 4.10:

$$\mathrm{gr}_j(U_q^\lambda) \to \Gamma(\mathrm{gr}_j(\mathcal{D}_q^\lambda)) \tag{4.11}$$

By Joseph $\mathrm{gr}(U_q^\lambda)$ is independent of λ. Also, $\Gamma(\mathrm{gr}_j(\mathcal{D}_q^\lambda))$ is independent of λ since it equals $\Gamma(\overline{p}^\star(\mathrm{gr}_j(M_\lambda)))$. Under these identifications the map 4.11 is independent of λ. Hence, 4.11 and so 4.7 are isomorphisms. $\square$.

Remark 4.6 1) *Note that the object in* $\mathbf{Proj}(A)$ *corresponding to* $\overline{p}^\star(U_q^{fin})$ *is* $A \otimes U_q^{fin}$. *This can be given the structure of an algebra* $A \star U_q^{fin}$. *Then one can see that our* $\mathcal{D}_q$*-modules becomes a category of objects in* $\mathbf{Proj}(A)$ *equipped with a graded action of this algebra and* λ*-compatibility. This relates our work to the work of Tanisaki, [T].*

2) *Differential operators on the big cell and its translates of quantum* G/B *gives the algebras of differential operators of Joseph, [J].*

4.4 Localization.

From now on we assume that q is not a root of unity.

Theorem 4.7 *For* $\lambda \in \mathfrak{h}^*$ *regular and dominant,* $\Gamma : \mathcal{D}^{\lambda}_{B_q}(G_q) \to \Gamma(\mathcal{D}^{\lambda}_q) - \mathrm{mod}$ *is an equivalence of categories.*

Our proof is very similar to Beilinson and Bernsteins proof of this theorem for classical flag-varieties.

In the following discussion V will denote a finite dimensional G_q-module. It is well known that V admits a filtration

$$0 \subset V_0 \subset \ldots \subset V_i \subset \ldots \subset V_n = V \tag{4.12}$$

of B_q-submodules where $V_i/V_{i-1} \cong k_{\mu_i}$ and $\mu_i > \mu_j \implies j > i$. (Thus μ_0 is the highest weight and μ_n the lowest weight of V.)

Lemma 4.8 *Let* $F \in \mathcal{M}_{B_q}(G_q)$ *and consider* $V \otimes F$ *as a* B_q*-module via the diagonal action.* (**a**) *We have* $F^{\dim V} \cong V \otimes F$ *as left* B_q*-modules.* **b**) *The filtration 4.12 induces a* B_q*-filtration*

$$0 \subset \ldots \subset V_i \otimes F \subset \ldots \subset V \otimes F \tag{4.13}$$

We have $V_i \otimes F/V_{i-1} \otimes F \cong k_{\mu_i} \otimes F \cong F \otimes k_{\mu_i} = F(-\mu_i)$ *as* B_q*-modules.*

Proof of lemma 4.8 (**a**) Let V^{triv} be V with the trivial B_q-action and $V^{triv} \otimes F(\cong F^{\dim V})$ the B_q-module with action on the second factor. The map

$$V \otimes F \to V^{triv} \otimes F;\ v \otimes f \to v_1 \otimes v_2 f \tag{4.14}$$

is a B_q-isomorphism. (Its inverse is $v \otimes f \to v_1 \otimes S(v_2)f$.)

(**b**) The only statement that needs a proof is that $k_\mu \otimes F \cong F \otimes k_\mu$. We construct an B_q-isomorphism $\pi : F \to k_\mu \otimes F \otimes k_{-\mu}$ by $\pi(f) = q^{-<\mu,\phi>} 1 \otimes f \otimes 1$ for $f \in F^\phi$. Here F^ϕ is the ϕ-weightspace of F. $\square$

The filtration 4.13 induces a projection and an injection

$$p_F : V \otimes F \to F(-\mu_n) \quad i_F : F \to V \otimes F(\mu_0)$$

respectively. (Here i_F is the inclusion $F(-\mu_0) \cong V_0 \otimes F \to V \otimes F$ tensored by $k_{-\mu_0}$.) The isomorphism $V \otimes F \cong F^{\dim V}$ induces an $\mathcal{O}_q$-module structure on $V \otimes F$ making it an object in $\mathcal{M}_{B_q}(G_q)$. With this structure the maps i_F and p_F are morphisms in $\mathcal{M}_{B_q}(G_q)$.

Remark 4.9 *Another way to see the $\mathcal{O}_q$-module structure on $V \otimes F$ is to define it as $\overline{p}^\star(V) \otimes_{\mathcal{O}_q} F$ as we did in section 3.6.*

Assume that $F \in \mathcal{D}^\lambda_{B_q}(G_q)$. Then each $V_i \otimes F$ becomes an U_q-module, by restricting the $\mathcal{D}_q$-action on F to an action of its subalgebra $U_q \cong 1 \otimes U_q$ and using the trivial U_q-action on V_i. In this case p_F and i_F are U_q-linear.

Lemma 4.10 *Assume that $F \in \mathcal{D}^\lambda_{B_q}(G_q)$.* (**a**) *If λ is dominant, then i_F has a splitting that is U_q and B_q-linear.* (**b**) *If λ is regular and dominant, then p_F has a splitting that is U_q and B_q-linear.*

Proof of lemma 4.10. **a**) The center $\mathcal{Z}$ of U_q acts on $V_i \otimes F(\mu_0)/V_{i-1} \otimes F(\mu_0)$ by the character $\chi_{-\lambda-\mu_0+\mu_i}$. But then $\chi_{-\lambda} \neq \chi_{-\lambda-\mu_0+\mu_i}$ for $i \neq 0$. Thus, by Harish-Chandra's theorem the map i_F splits U_q-linearly. The compatibility of the U_q and B_q-actions implies that the splitting map is B_q linear as well.

(**b**) The center $\mathcal{Z}$ of U_q acts on $V_i \otimes F/V_{i-1} \otimes F$ by the character $\chi_{-\lambda+\mu_i}$. But then $\chi_{-\lambda+\mu_n} \neq \chi_{-\lambda+\mu_i}$ for $i \neq 0$. Again, this implies that the map p_F splits U_q-linearly and hence B_q-linearly. $\square$

Remark 4.11 *Exactly as in the classical theory the splittings of i_F and of p_F given by lemma 4.10 are not $\mathcal{O}_q$-linear. Since the maps are B_q-linear and since lemma 3.9 shows that the cohomologies $R^j\Gamma$ can be computed by taking injective resolutions of underlying B_q-modules, we see that they induce splittings on cohomologies.*

Proof of theorem 4.7

i) The functor Γ is exact. Let $F \in \mathcal{D}^\lambda_{B_q}(G_q)$. We must prove that $R^j\Gamma(F) = 0$. This will follow if we can prove that for any noetherian $M \in \mathcal{M}_{B_q}(G_q)$ and

injection $M \hookrightarrow F$ in $\mathcal{M}_{B_q}(G_q)$, the induced maps $a : R^j\Gamma(M) \to R^j\Gamma(F)$ is the zero map for all $j \geq 0$.

Let V be as in lemma 4.12. Assume that μ_0 is sufficiently large for $R^j\Gamma(M(\mu_0)) = 0$ to hold. We get a commutative diagram

$$\begin{array}{ccc} R^j\Gamma(M) & \stackrel{i_M}{\to} & R^j\Gamma(V \otimes M(\mu_0)) \\ \downarrow_a & & \downarrow \\ R^j\Gamma(F) & \stackrel{i_F}{\to} & R^j\Gamma(V \otimes F(\mu_0)) \end{array} \tag{4.15}$$

Since $R^j\Gamma(V \otimes M(\mu_0)) \cong R^j\Gamma(M^{\dim V}(\mu_0)) = 0$, the composition $i_F \circ a$ is zero. Since i_F has a section by lemma 4.13, a is zero.

ii) The functor Γ *is an equivalence of categories.* Since we know that Γ is exact this follows from general considerations if we can prove that any $F \in \mathcal{D}^{\lambda}_{B_q}(G_q)$ satisfies $\Gamma(F) \neq 0$.

Since p_F splits, $\Gamma(F(-\mu_n))$ is a direct summand in $\Gamma(V \otimes F) \cong \Gamma(F)^{\dim V}$. If μ_n is sufficiently negative we have $\Gamma(F(-\mu_n)) \neq 0$. Hence, $\Gamma(F) \neq 0$. $\square$

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